

HARMONIC-OSCILLATOR-BASED EFFECTIVE THEORY

W. C. HAXTON

*Inst. for Nuclear Theory and Dept. of Physics, University of Washington
Seattle, WA 98195, USA*

E-mail: haxton@phys.washington.edu

I describe harmonic-oscillator-based effective theory (HOBET) and explore the extent to which the effects of excluded higher-energy oscillator shells can be represented by a contact-gradient expansion in next-to-next-to-leading order (NNLO). I find the expansion can be very successful provided the energy dependence of the effective interaction, connected with missing long-wavelength physics associated with low-energy breakup channels, is taken into account. I discuss a modification that removes operator mixing from HOBET, simplifying the task of determining the parameters of an NNLO interaction.

Keywords: Nuclear structure; Effective theory; NNLO interactions

1. Introduction

Often the problem of calculating long-wavelength nuclear observables – binding energies, radii, or responses to low-momentum probes – is formulated in terms of pointlike, nonrelativistic nucleons interacting through a potential. To solve this problem theorists have developed both nuclear models, which are not systematically improvable, and exact numerical techniques, such as fermion Monte Carlo. Because the nuclear many-body problem is so difficult – one must simultaneously deal with anomalously large NN scattering lengths and a potential that has a short-range, strongly repulsive core – exact approaches are numerically challenging, so far limited to the lighter nuclei within the 1s and lower 1p shells. The Argonne theory group has been one of the main developers of such exact methods.¹

However, effective theory (ET) offers an alternative, a method that limits the numerical difficulty of a calculation by restricting it to a finite Hilbert space (the P - or “included”-space), while correcting the bare Hamiltonian H (and other operators) for the effects of the Q - or “excluded”-space. Calculations using the effective Hamiltonian H^{eff} within P reproduce the results using H within $P + Q$, over the domain of overlap. That is, the effects of Q

on P -space calculations are absorbed into $P(H^{eff} - H)P$. There may exist some systematic expansion – perhaps having to do with the shorter range of interactions in Q – that simplifies the determination $P(H^{eff} - H)P$.^{2,3}

One interesting challenge for ET is the case of a P -space basis of harmonic oscillator (HO) Slater determinants. This is a special basis for nuclear physics because of center-of-mass separability: if all Slater determinants containing up to n oscillator quanta are retained, H^{eff} will be translationally invariant (assuming H is). Such bases are also important because of powerful shell-model (SM) techniques that have been developed for interactive diagonalization and for evaluating inclusive responses. The larger P can be made, the smaller the effects of $H^{eff} - H$.

There are two common approaches to the ET problem. One is the determination of $P(H^{eff} - H)P$ from a given H known throughout $P + Q$, a problem that appears naively to be no less difficult than the original $P + Q$ diagonalization of H . However, this may not be the case if H is somehow simpler when acting in Q . For example, if Q contains primarily high-momentum (short-distance) interactions, then $H^{eff} - H$ might have a cluster expansion: it becomes increasingly unlikely to have m nucleons in close proximity, as m increases (e.g., a maximum of four nucleons can be in a relative s-state). Thus one could approximate the full scattering series in Q by successive two-body, three-body, etc., terms, with the expectation that this series will converge quickly with increasing nucleon number. This would explain why simple two-nucleon ladder sums – g-matrices – have been somewhat successful as effective interactions⁴ (however, see Ref. 5,6).

The second approach is that usually taken in effective field theories,³ determining H^{eff} phenomenologically. This is the “eliminate the middleman” approach: H itself is an effective interaction, parameterized in order to reproduce NN scattering and other data up to some energy. So why go to the extra work of this intermediate stage between QCD and SM-like spaces? This alternative approach begins with PHP , the long-range NN interaction that is dominated by pion exchange and constrained by chiral symmetry. The effects of the omitted Q -space, $P(H^{eff} - H)P$, might be expressed in some systematic expansion, with the coefficients of that expansion directly determined from data, rather than from any knowledge of H acting outside of P .

While we explored this second approach some years ago, some subtle issues arose, connected with properties of HO bases. This convinced us that the first step in our program should be solving and thoroughly understanding the effective interactions problem via the first approach, so that we

would have answers in hand to test the success of more phenomenological approaches. Thus we proceeded to follow the first approach using a realistic NN potential, *av18*,⁷ generating H^{eff} numerically for the two- and three-body problems in a variety of HO SM spaces. Here I will use these results to show that a properly defined short-range interaction provides an excellent representation of the effective interaction. This is an encouraging result, one that suggests a purely phenomenological treatment of the effective interaction might succeed.

The key observation is that HOBET is an expansion around momenta $k \sim 1/b$, and thus differs from EFT approaches that expand around $k \sim 0$. Consequently a HOBET P -space lacks both high-momentum components important to short-range NN interactions and long-wavelength components important to minimizing the kinetic energy. While our group has previously discussed some of the consequences of the combined infrared/ultraviolet problem in HOBET, here I identify another: a sharp energy dependence in H^{eff} that must be addressed before any simple representation of $H^{eff} - H$ is possible. This, combined with a trick to remove operator mixing, leads to a simple and successful short-range expansion for $H^{eff} - H$. I conclude by noting how these results may set the stage for a successful determination of the HOBET H^{eff} directly from data.

2. Review of the Bloch-Horowitz Equation

The basis for the approach described here is the Bloch-Horowitz (BH) equation, which generates a Hermitian, energy-dependent effective Hamiltonian, H^{eff} , which operates in a finite Hilbert space from which high-energy HO Slater determinants are omitted:

$$H^{eff} = H + H \frac{1}{E - QH} QH$$

$$H^{eff} |\Psi_P\rangle = E |\Psi_P\rangle \quad |\Psi_P\rangle = (1 - Q) |\Psi\rangle. \quad (1)$$

Here H is the bare Hamiltonian and E and Ψ are the exact eigenvalue and wave function (that is, the results of a full solution of the Schroedinger equation for H in $P+Q$). The BH equation must be solved self-consistently, as H^{eff} depends on E . If this is done, the model-space calculation reproduces the exact E , and the model-space wave function Ψ_P is simply the restriction of Ψ to P . If one takes for P a complete set of HO Slater determinants with HO energy $\leq \Lambda_P \hbar \omega$, H^{eff} will be translational invariant. P is then defined by two parameters, Λ_P and the HO size parameter b .

The BH equation was solved numerically for the *av18* potential using

two numerical techniques. In work carried out in collaboration with C.-L. Song,⁸ calculations were done for the deuteron and $^3\text{He}/^3\text{H}$ by directly summing the effects of $av18$ in the Q -space. Because this potential has a rather hard core, sums to $140 \hbar\omega$ were required to achieve ~ 1 keV accuracy in the deuteron binding energy, and $70 \hbar\omega$ to achieve ~ 10 keV accuracies for $^3\text{He}/^3\text{H}$. In work carried out with T. Luu,⁹ such cutoffs were removed by doing momentum-space integrations over all possible excitations.

The results are helpful not only to the goals discussed previously, but also in illustrating general properties of H^{eff} that may not be widely appreciated. For example, Table 1 gives the evolution of the P -space ^3He $av18$ wave function as a function of increasing Λ_P for fixed b . Ψ_P evolves simply, with each increment of Λ_P adding new components to the wave function, while leaving previous components unchanged. One sees that the probability of residing in the model space grows slowly from its $0\hbar\omega$ value (31%) toward unity.

Table 1. Evolution of the ^3He $av18$ HO wave function Ψ_P with Λ_P

state	amplitude					
	$0\hbar\omega$	$2\hbar\omega$	$4\hbar\omega$	$6\hbar\omega$	$8\hbar\omega$	exact
	(31.1%)	(57.4%)	(70.0%)	(79.8%)	(85.5%)	(100%)
$ 0, 1\rangle$	0.5579	0.5579	0.5579	0.5579	0.5579	0.5579
$ 2, 1\rangle$	0.0000	0.0463	0.0461	0.0462	0.0462	0.0463
$ 2, 2\rangle$	0.0000	-0.4825	-0.4824	-0.4824	-0.4824	-0.4826
$ 2, 3\rangle$	0.0000	0.0073	0.0073	0.0073	0.0073	0.0073
$ 4, 1\rangle$	0.0000	0.0000	-0.0204	-0.0204	-0.0204	-0.0205
$ 4, 2\rangle$	0.0000	0.0000	0.1127	0.1127	0.1127	0.1129
$ 4, 3\rangle$	0.0000	0.0000	-0.0419	-0.0420	-0.0421	-0.0423

Effective operators are defined by

$$\hat{O}^{eff} = (1 + HQ \frac{1}{E_f - HQ}) \hat{O} (1 + \frac{1}{E_i - QH} QH) \quad (2)$$

and must be evaluated between wave functions Ψ_P having the nontrivial normalization illustrated in Table 1 and determined by

$$1 = \langle \Psi | \Psi \rangle = \langle \Psi_P | \hat{1}^{eff} | \Psi_P \rangle. \quad (3)$$

The importance of this is illustrated in Fig. 1, where the elastic magnetic responses for deuterium and ^3He are first evaluate with exact wave functions Ψ_P but bare operators, then re-evaluated with the appropriate effective

operators. Bare operators prove a disaster even at intermediate momentum transfers of $2\text{--}3\text{ f}^{-1}$. By using the effective operator and effective wave function appropriate to Λ_P , the correct result – the form factor is independent of the choice of Λ_P (or b) – is obtained, as it must in any correct application of effective theory.

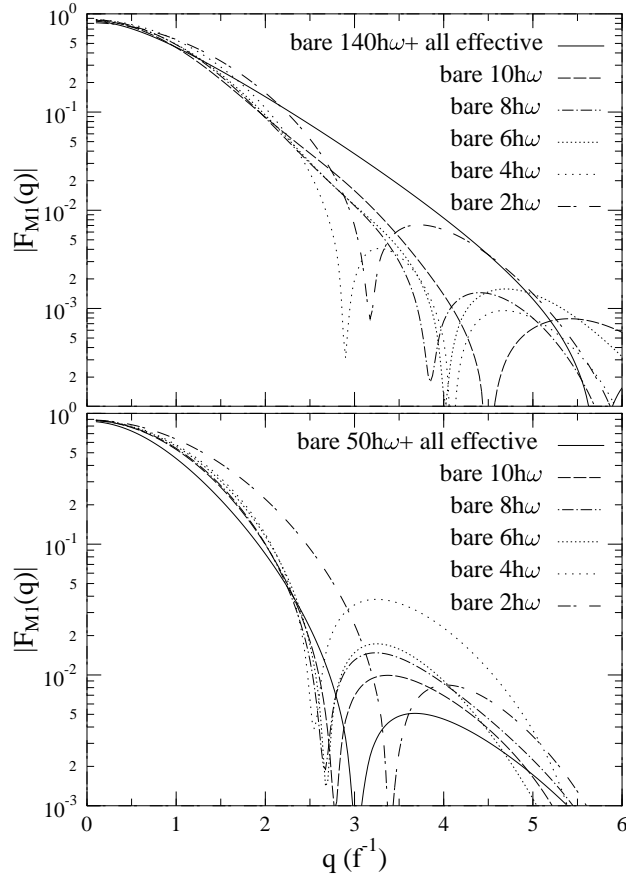


Fig. 1. Deuteron and ${}^3\text{He}$ elastic magnetic form factors evaluated for various P -spaces with bare operators (various dashed and dotted lines) and with the appropriate effective operators (all results converge to the solid lines).

The choice of a HO basis excludes not only high-momentum components of wave functions connected with the hard core, but also low-momentum components connected with the proper asymptotic fall-off of the tail of

the wave functions. This combined ultraviolet/infrared problem was first explored by us in connection with the nonperturbative behavior of H^{eff} : the need to simultaneously correct for the missing long- and short-distance behavior of Ψ_P is the reason one cannot define a simple P that makes evaluation of H^{eff} converge rapidly. We also found a solution to this problem, a rewriting of the BH equation in which the relative kinetic energy operator is summed to all orders. This summation can be viewed as a transformation of a subset of the Slater determinants in P to incorporate the correct asymptotic falloff. This soft physics, obtained from an infinite sum of high-energy HO states in Q , is the key to making H^{eff} perturbative.

It turns out this physics is also central to the issue under discussion here, the existence of a simple representation for $H^{eff} = H + H \frac{1}{E-QH} QH$, where $H = T + V$. The reorganized BH H^{eff} is the sum of the three left-hand-side (LHS) terms in Eqs. (4)

$$\begin{aligned} \langle \alpha | T + TQ \frac{1}{E-QT} QT | \beta \rangle &\xrightarrow{\text{nonedge}} \langle \alpha | T | \beta \rangle \\ \langle \alpha | \frac{E}{E-TQ} V \frac{E}{E-QT} | \beta \rangle &\xrightarrow{\text{nonedge}} \langle \alpha | V | \beta \rangle \\ \langle \alpha | \frac{E}{E-TQ} V \frac{1}{E-QH} QV \frac{E}{E-QT} | \beta \rangle &\xrightarrow{\text{nonedge}} \langle \alpha | V \frac{1}{E-QH} QV | \beta \rangle \end{aligned} \quad (4)$$

The first LHS term is the effective interaction for T , the relative kinetic energy. As QT acts as a ladder operator in the HO, $E/E-QT$ is the identity except when it operates on an $|\alpha\rangle$ with energy $\Lambda_P \hbar \omega$ or $(\Lambda_P - 1) \hbar \omega$. We will call these Slater determinants the edge states. For nonedge states, this new expression and the BH form given in Eq. (1) both reduce to the expressions on the right of Eqs. (4).

Noting that the first LHS term in Eqs. (4) can be rewritten as

$$\langle \alpha | \frac{E}{E-TQ} (T - \frac{TQT}{E}) \frac{E}{E-QT} | \beta \rangle, \quad (5)$$

we see that the QT summation can be viewed as a transformation to a new basis for P , $E/(E-QT)|\alpha\rangle$, that is orthogonal but not orthonormal. This edge-state basis builds in the proper asymptotic behavior governed by QT (free propagation) and the binding energy E . The transformation preserves translational invariance, as T is the relative kinetic energy operator. Viewed in the transformed basis, the appropriate effective interaction is given by the LHS terms in the square brackets in Eqs. (6) below.

Alternatively, the results can be viewed as two equivalent expressions for the effective interaction between HO states, but with a different division

between “bare” and “rescattering” contributions

$$\begin{aligned} \text{bare} : & \frac{E}{E - TQ} \left[H - \frac{TQT}{E} \right] \frac{E}{E - QT} \Leftrightarrow H \\ \text{rescattering} : & \frac{E}{E - TQ} \left[V \frac{1}{E - QH} QV \right] \frac{E}{E - QT} \Leftrightarrow H \frac{1}{E - QH} QH \quad (6) \end{aligned}$$

It is this new division that is critical. The expressions are identical for nonedge states. But for edge states, only the expression on the left isolates a quantity, VGV , that is short-range and nonperturbative. We will see that this is the term that can be represented by a simple, systematic expansion.

Figure 2 shows the extended tail that is induced by $E/E - QT$ acting on a HO state. Figure 3 is included to emphasize that there are important numerical differences between the two expressions in Eqs. (6). It compares calculations done for the deuteron using the two “bare” interactions: thus in both cases V enters only linearly between low-momentum states, and all multiple scattering of V in Q is ignored. Figure 3 gives the resulting deuteron binding energy as a function of b , for several values of Λ_P . For the standard form of the BH equation, a small model space overestimates the kinetic energy (too confining) and overestimates short-range contributions to V (too little freedom to create the needed wave-function “hole”). Making b larger to lower the kinetic energy exacerbates the short-range problem, and conversely. Thus the best b is a poor compromise that, even in a $10 \hbar\omega$ bare calculation, fails to bind the deuteron. But the new bare H on the LHS of Eqs. (6) sums QT to give the correct wave-function behavior at large r , independent of b . Then, for the choice $b \sim 0.4\text{--}0.5$ f, the short-range physics can be absorbed directly into the P space. The result is excellent 0th-order ground-state energy, with the residual effects of multiple scattering through QV being very small and perturbative.⁹

3. Harmonic-Oscillator-Based Effective Theory

Now I turn to the question of whether (and how) the Q -space rescattering contribution to H^{eff} might be expressed through some systematic short-range expansion. There are two steps important in applying such an expansion to HOBET. One has to do with the form of the short-range expansion. A contact-gradient (CG) expansion, constructed to include all possible LO (leading order), NLO (next-to-leading order), NNLO (next-to-next-to-leading order), ..., interactions is commonly used,

$$\begin{aligned} & a_{LO}^{ss}(\Lambda_P, b) \delta(\mathbf{r}) + a_{NLO}^{ss}(\Lambda_P, b) (\overleftarrow{\nabla}^2 \delta(\mathbf{r}) + \delta(\mathbf{r}) \overrightarrow{\nabla}^2) + \\ & a_{NNLO}^{ss,22}(\Lambda_P, b) \overleftarrow{\nabla}^2 \delta(\mathbf{r}) \overrightarrow{\nabla}^2 + a_{NNLO}^{ss,40}(\Lambda_P, b) (\overleftarrow{\nabla}^4 \delta(\mathbf{r}) + \delta(\mathbf{r}) \overrightarrow{\nabla}^4). \quad (7) \end{aligned}$$

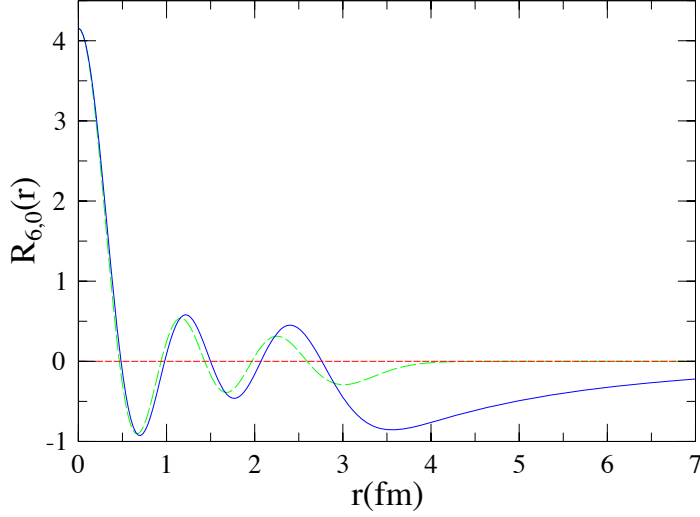


Fig. 2. A comparison of the $|nl\rangle$ and extended $(E/E - QT)|nl\rangle$ radial wave functions, for the edge state $(n, l) = (6, 0)$ in a $\Lambda_P = 10$ deuteron calculation. Note that the normalization of the extended state has been adjusted to match that of $|nl\rangle$ at $r=0$, in order to show that the shapes differ only at large r . Thus the depletion of the extended state at small r is not apparent in this figure.

Because HOBET is an expansion around a typical momentum scale $\sim 1/b$, rather than around $\vec{k} = 0$, it is helpful to redefine the derivatives appearing in the CG expansion. Noting

$$\vec{\nabla}^n \exp i\vec{k} \cdot \vec{r} |_{k=0} = 0, n = 1, 2, \dots, \quad (8)$$

we demand by analogy in HOBET

$$\vec{\nabla}^n \psi_{1s}(b) = 0, n = 1, 2, \dots \quad (9)$$

This can be accomplished by redefining the operators \hat{O} of Eqs. (7) by

$$\hat{O} \rightarrow e^{r^2/2} \hat{O} e^{r^2/2} \quad (10)$$

The gradients in Eq. (7) then act on polynomials in r , a choice that removes all operator mixing. That is, if a_{LO} is fixed in LO to the $n = 1$ to $n = 1$ matrix element, where n is the nodal quantum number, it remains fixed in NLO, NNLO, etc. Furthermore, the expansion is in nodal quantum numbers, e.g.,

$$\vec{\nabla}^2 \sim (n-1) \quad \vec{\nabla}^4 \sim (n-1)(n-2) \quad (11)$$

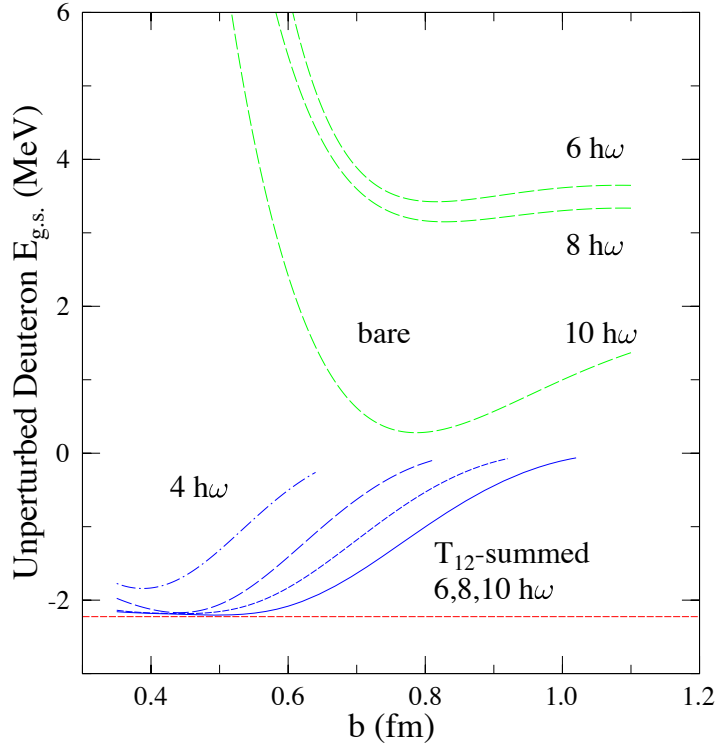


Fig. 3. Deuteron ground-state convergence in small P -spaces, omitting all effects due to the multiple scattering of V in Q . The standard BH formulation with $P(T + V)P$ fails to bind the deuteron, even with $\Lambda_P = 10$. The reorganized BH equation, where QT has been summed to all orders but V still appears only linearly, reproduces the correct binding energy for $\Lambda_P=6$.

so that matrix elements become trivial to evaluate in any order. It can be shown that the leading order in n contribution agrees with the plane-wave result, and that operator coefficients are a generalization of standard Talmi integrals for nonlocal potentials, e.g.,

$$a_{NNLO}^{ss,22} \sim \int_0^\infty \int_0^\infty e^{-r_1^2} r_1^2 V(r_1, r_2) r_2^2 e^{-r_2^2} r_1^2 r_2^2 dr_1 dr_2 \quad (12)$$

The next step is to identify that quantity in the BH equation that should be identified with the CG expansion. This has to do with the two forms of the BH equation discussed previously. Consider the process of progressively

integrating out Q in favor of the CG expansion, beginning at $\Lambda \gg \Lambda_P$ and progressing to $\Lambda = \Lambda_P$. Using the projection operator

$$Q_\Lambda = \sum_{\alpha=\Lambda_P+1}^{\Lambda} |\alpha\rangle\langle\alpha| \quad Q_{\Lambda_P} \equiv 0 \quad (13)$$

we can isolate the contributions, above some scale Λ , to the two BH rescattering terms of Eqs. (6)

$$\begin{aligned} \Delta(\Lambda) &= H \frac{1}{E - QH} QH - H \frac{1}{E - Q_\Lambda H} Q_\Lambda H \\ \Delta_{QT}(\Lambda) &= \frac{E}{E - TQ} \left[V \frac{1}{E - QH} QV - V \frac{1}{E - Q_\Lambda H} Q_\Lambda V \right] \frac{E}{E - QT}. \end{aligned} \quad (14)$$

The goal of a CG expansion might be successful reproduction of the matrix elements of $\Delta(\Lambda)$ – the Q -space rescattering contributions for the standard form of the BH equation – as $\Lambda \rightarrow \Lambda_P$. This would allow us to replace all Q -space rescattering by a systematic short-range expansion, opening the door to a purely phenomenological determination of H^{eff} for the SM. The test case will be an $8\hbar\omega$ P -space calculation for the deuteron ($\Lambda_P = 8$, $b=1.7$ f). The running of the 15 independent 3S_1 matrix elements of $\Delta(\Lambda)$ are plotted in Fig. 4a. Five of these are distinguished because they involve an edge-state bra or ket (or both). The evolution of these contributions with Λ is seen to be somewhat less regular than that of nonedge-state matrix elements. The results for $\Lambda = \Lambda_P$ show that rescattering is responsible for typically 12 MeV of binding energy.

The CG fit to the results in Fig. 4a were done in LO, NLO, and NNLO as a function of Λ , using the standard form of the BH equation. The coefficients are fit to the lowest-energy matrix elements. Thus in LO $a_{LO}(\Lambda)$ is fixed by the $1s - 1s$ matrix element, leaving 14 unconstrained matrix elements; the NNLO fit ($1s - 1s$, $1s - 2s$, $1s - 3s$, and $2s - 2s$) leaves 11 matrix elements unconstrained. This is easily done, because the operators do not mix; e.g., among these four, only the $1s - 3s$ matrix element is influenced by $a_{NNLO}^{ss,40}$. The result is a set of coefficient that run as a function of Λ in the usual way, with a_{LO} small and dominant for large Λ , and with the NLO and NNLO terms turning on as the scale is dropped. Figs. 4b-d show the residuals – the differences between the predicted and calculated matrix elements. For non-edge-state matrix elements the scale at which typical residuals in Δ are significant, say greater than 100 keV (above $\sim 1\%$), is brought down successively, e.g., from $\sim 100\hbar\omega$, to $\sim 55\hbar\omega$ (LO), to $\sim 25\hbar\omega$ (NLO), and finally to $\sim \Lambda_P\hbar\omega$ (NNLO). But matrix elements involving

edge states break this pattern: the improvement is not significant, with noticeable deviations remaining at $\sim 100\hbar\omega$ even at NNLO.

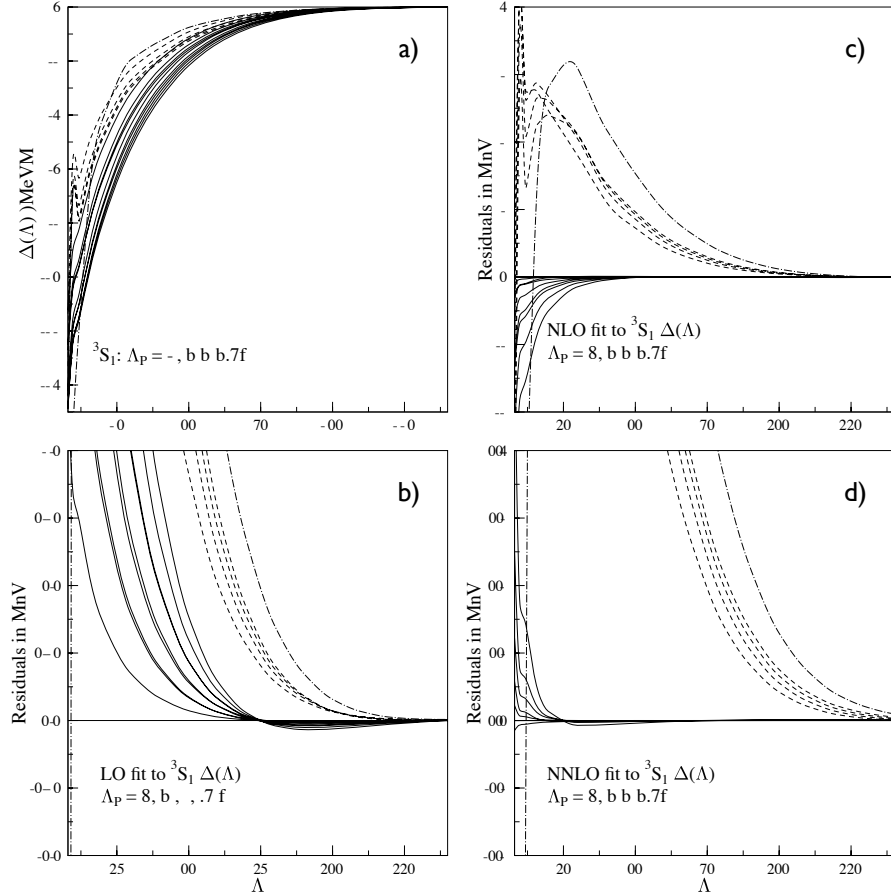


Fig. 4. In a) rescattering contributions to H^{eff} due to excitations in Q above Λ are given for the standard form of the BH equation. These results are for the 15 matrix elements that arise in an $8\hbar\omega$ calculation for the deuteron, with $b=1.7$ f. In b)-d) the residuals of LO, NLO, and NNLO fits are shown (see text). Matrix elements with bra or ket (dashed) or both (dot-dashed) edge states are seen not to improve systematically.

This failure could be anticipated: because QT strongly couples nearest shells across the $P-Q$ boundary, $H \frac{1}{E-QH} QH$ contains long-range physics. The candidate short-range interaction is $V \frac{1}{E-QH} QV$, not $H \frac{1}{E-QH} QH$: this

is the reason QT should be first summed, putting the BH equation in a form – the LHS of Eqs. (6) – that isolates this quantity. This reorganization affects edge-state matrix elements only, those with the large residuals in Figs. 4b-d.

To use the reorganized BH equation, the QT sums appearing in Eqs. (4) must be completed. There are several procedures for doing this, but one convenient method exploits the raising/lowering properties of T . The result is a series of continued fractions $\tilde{g}_i(2E/\hbar\omega, \{\alpha_i\}, \{\beta_i\})$, where $\alpha_i = (2n+2i+l-1/2)/2$ and $\beta_i = \sqrt{(n+i)(n+i+l+1/2)}/2$. For any operator \hat{O} (e.g., V , $V \frac{1}{E-QH} QV$, etc.)

$$\langle n'l' | \frac{E}{E-TQ} O \frac{E}{E-QT} | nl \rangle = \sum_{i,j=0} \tilde{g}_j(n', l') \tilde{g}_i(n, l) \langle n'+j | l | O | n+i | l \rangle \quad (15)$$

It follows that the coefficients of the CG expansion for a HO basis must be redefined for edge states, with a state- and E-dependent renormalization

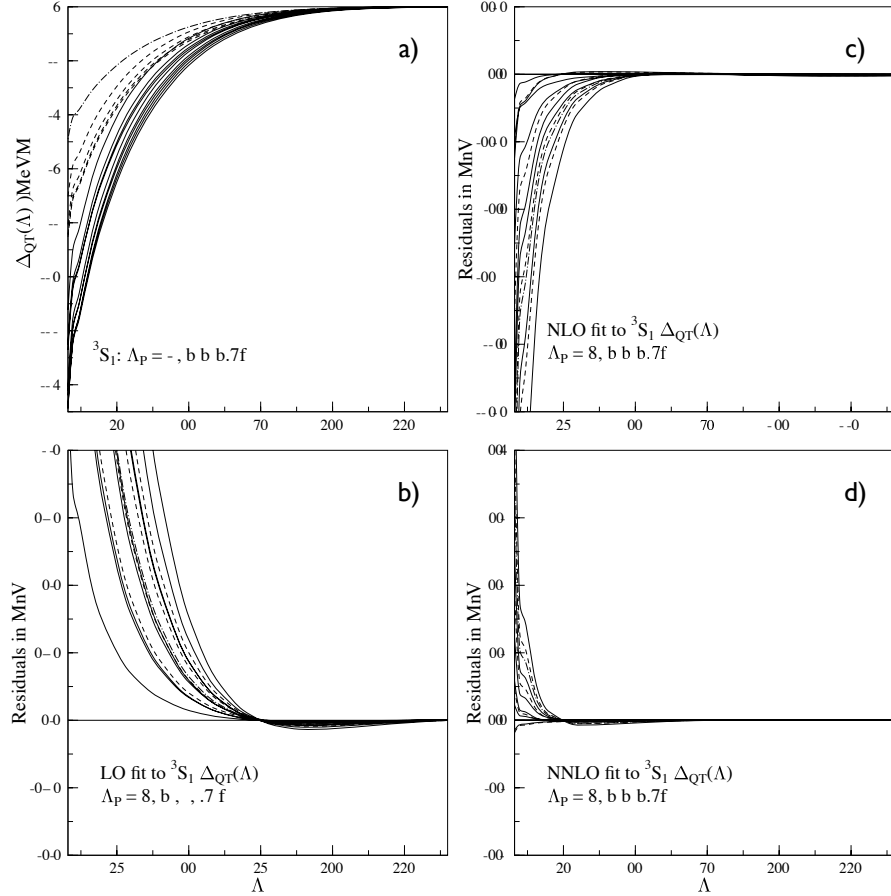
$$a_{LO} \rightarrow a'_{LO}(E/\hbar\omega; n', l', n, l) = a_{LO} \sum_{i,j=0} \tilde{g}_j(n', l') \tilde{g}_i(n, l) \times \left[\frac{\Gamma(n'+j+1/2)\Gamma(n+i+1/2)}{\Gamma(n'+1/2)\Gamma(n+1/2)} \right]^{1/2} \left[\frac{(n'-1)!(n-1)!}{(n'+j-1)!(n+i-1)!} \right]^{1/2} \quad (16)$$

This renormalization, which introduces no new parameters, can be evaluated in a similar way for heavier systems: T remains a raising operator.

Figs. 5 shows the results: the difficulties encountered for $\Delta(\Lambda)$ do not arise for $\Delta_{QT}(\Lambda)$. The edge-state matrix elements are now well behaved, and the improvement from LO to NLO to NNLO is systematic in all cases. When $\Lambda \rightarrow \Lambda_P$, the CG potential continues to reproduce H^{eff} for the $av18$ potential remarkably well, with an the average error in 3S_1 NNLO matrix elements of about 100 keV (or 1% accuracy). Other channels we explored behaved even better: the average error for the 15 1S_0 matrix elements is about 10 keV (or 0.1% accuracy). Because all matrix elements of H^{eff} are reproduced well, the CG potential preserves spectral properties, not simply properties of the lowest energy states within P . The NNLO calculation in the 3S_1 - 3D_1 channel yields a deuteron ground-state energy of -2.21 MeV.

Several points can be made:

- The net effect of the QT summation is to weaken the CG potential for HO edge states: the resulting, more extended state has a reduced probability at small r . Consequently the effects of QV are weaker than in states immune from the effects of QT .
- The very strong QT coupling of the P and Q spaces is clearly problematic



sion with fixed coefficients can be used between these states, following the reorganized BH equation of Eqs. (4).

- While our calculations have been limited to the deuteron, this same phenomena must arise in heavier systems treated in HO bases – QT remains the ladder operator. As the issue is extended states that minimize the kinetic energy, it is clear that the relevant parameter must be the Jacobi coordinate associated with the lowest breakup channel. This could be an issue for treatments of H^{eff} based on the Lee-Suzuki transformation, which transforms the interaction into an energy-independent one. In approaches like the no-core shell model,¹⁰ the Lee-Suzuki transformation is generally not evaluated exactly, but instead only at the two-body level. If such an approach were applied, for example, to ${}^6\text{Li}$, a system weakly bound (1.475 MeV) to breaking up as $\alpha+d$, it is not obvious that a two-body Lee-Suzuki transformation would treat the relevant Jacobi coordinate responsible for the dominant energy dependence. This should be explored.
- I believe the conclusions about the CG expansion will apply to other effective interactions. For example, V-low-k,¹¹ a soft potential obtained by integrating out high-momentum states, is derived in a plane-wave basis, where T is diagonal. Thus it should be analogous to our CG interaction, requiring a similar renormalization when embedded into a HO SM space. It would be interesting to test this conclusion.

4. Summary

These results show that the effective interaction in the HO SM must have a very sharp dependence on the binding energy, defined as the energy of the bound state relative to the first open channel. This is typically 0 to 10 MeV for the bound states of most nuclei (and 2.22 MeV for the deuteron ground state explored here). Once this energy dependence is identified, the set of effective interaction matrix elements can be represented quite well by a CG expansion, and the results for successive LO, NLO, and NNLO calculations improve systematically.

This result suggests that the explicit energy dependence of the BH equation is almost entirely due to QT – though this inference, based on the behavior of matrix elements between states with different number of HO quanta, must be tested in a case where multiple bound states exist.

We also presented a simple redefinition of the gradients associated with CG expansions, viewing the expansion as one around a momentum scale $\sim 1/b$. This definition removes operator mixing, making NNLO and higher-order fits very simple. The expansion then becomes one in nodal quantum

numbers, with the coefficients of the expansion related to Talmi integrals, generalized for nonlocal interactions.

While our exploration here has been based on “data” obtained from an exact BH calculation of the effective interaction for the *av18* potential, this raises the question, is such a potential necessary to the SM? That is, now that the success of an NNLO description of Q -space contributions is established, could one start with *PHP* and determine the coefficients for such a potential directly from data, without knowledge of matrix elements of H outside of P ? I believe the answer is yes, even in cases (like the deuteron) when insufficient information is available from bound states. It turns out that the techniques described here can be extended into the continuum, so that observables like phase shifts could be combined with bound-state information to determine the coefficients of such an expansion. An effort of this sort is in progress.

I thank M. Savage for helpful discussions, and T. Luu and C.-L. Song for enjoyable collaborations. This work was supported by the U.S. Department of Energy Division of Nuclear Physics and by DOE SciDAC grant DE-FG02-00ER-41132.

5. References

References

1. S. C. Pieper and R. B. Wiringa, *Ann.Rev.Nucl.Part.Sci.* **51**, 53 (2001).
2. S. Weinberg, *Phys. Lett.* **B251**, 288 (1990), *Nucl. Phys.* **B363**, 3 (1991), and *Phys. Lett.* **295**, 114 (1992).
3. S. R. Beane, P. F. Bedaque, W. C. Haxton, D. R. Phillips, and M. J. Savage, in *At the Frontier of Particle Physics, Vol. 1* (World Scientific, Singapore, 2001), p. 133.
4. T. T. S. Kuo and G. E. Brown, *Nucl. Phys.* **A114**, 241 (1968).
5. B. R. Barrett and M. W. Kirson, *Nucl. Phys.* **A148**, 145 (1970).
6. T. H. Shucan and H. A. Wiedenmüller, *Ann. Phys. (N.Y.)* **73**, 108 (1972) and **76**, 483 (1973).
7. R. B. Wiringa, V. Stoks, and R. Schiavilla, *Phys. Rev.* **C51**, 28 (1995).
8. W. C. Haxton and C.-L. Song, *Phys. Rev. Lett.* **84**, 5484 (2000).
9. W. C. Haxton and T. C. Luu, *Nucl. Phys.* **A690**, 15 (2001) and *Phys. Rev. Lett.* **89**, 182503 (2002); T.C. Luu, S. Bogner, W. C. Haxton, and P. Navratil, *Phys. Rev.* **C70**, 014316 (2004).
10. P. Navratil, J. P. Vary, B. R. Barrett, *Phys. Rev. Lett.* **84**, 5728 (2000).
11. A. Schwenk, G. E. Brown, and B. Friman, *Nucl. Phys.* **A703**, 745 (2002); A. Schwenk, *J. Phys.* **G31**, S1273 (2005).